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1993 J. Phys. A: Math. Gen. 26 6285

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J. Phys. A: Math. Gen. 26 (1993) 6285–6301. Printed in the UK

Quantum kinematics and boson ladder operators of non-Abelian non-compact Lie groups

J Krause

Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

Received 5 January 1993

Abstract. Quantum kinematics is revisited, as a group-theoretic quantization procedure within the regular representation of non-Abelian non-compact r-dimensional Lie groups. The set of r basic quantum-kinematic invariant operators is exhibited; generalized Heisenberg commutation relations and the structure of the closed generalized Weyl–Heisenberg algebra of the quantized group are also discussed. Then it is shown how these structures yield a complete set of r 'annihilation' and 'creation' boson operators, which give rise to several intrinsic (i.e. embedded) Lie algebras, obtained in the standard way, within the quantized group model. As a miscellaneous example, these features are discussed within the quantum-kinematic theory of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$, and some interesting *possibilities for elementary particle theory are conjectured in the light of the attained results.*

1. Introduction

In a previous paper a formalism of non-Abelian group quantization was discussed, within the regular representation of non-compact Lie groups (Krause 1991, henceforth referred to as paper I). It was shown that all such r-dimensional groups have a set of r basic quantum-kinematic invariant operators, which substantially differ from the Casimir invariant operators of the traditional theory of Lie algebras and their enveloping algebras. The relation of the conventional invariants with the new quantum-kinematic invariants was also examined in that paper.

The importance of Lie group invariant operators is well known, both from the mathematical point of view as well as for their physical applications. However, let us here only remark that some Lie groups have no Casimir operator, *sensu stricto*; other Lie groups have only transcendental invariant operators that do not belong to the enveloping algebra. Moreover, it also happens that some Lie groups have no traditional invariant operators at all.

The novelty introduced by our study is immediate, if one observes that hitherto all invariant operators of Lie group theory have been defined as functions of the generators that commute with all the generators of a given representation (cf Barut and Raczka 1977). For the sake of briefness we here refer to this current notion as the traditional invariants of Lie group theory.

As was shown in an earlier paper, one arrives at completely different results if one uses the group quantization method (Krause 1985), for then it turns out that every r-dimensional Lie group has a set of r basic quantum-kinematic invariant operators,

0305-4470/93/226285+17$07.50 © 1993 IOP Publishing Ltd

which are defined as functions of the generators and of the generalized position operators (cf below) that commute with all the generators. Moreover, once a Lie group has been 'quantized', its quantum-kinematic invariant operators arise in a rather natural manner (even in those extreme cases where the group has no traditional invariant at all). In paper I we prove this fact for a special kind of non-compact Lie group. Though this feature is valid for all kinds of Lie groups (whether compact or non-compact), quantum kinematics of compact Lie groups, in general, sets a rather difficult issue. (The consideration of the general formalism of quantum kinematics for compact Lie groups is postponed to some forthcoming papers.) As for the physical motivation of non-Abelian quantum kinematics the reader is referred to previous work on this subject and to the literature quoted therein.

In this paper this subject is studied further. In particular, here a new kind of quantum-kinematic 'annihilation' and 'creation' boson-operators associated with non-Abelian non-compact Lie groups are introduced, which may have some interesting physical applications.

As a matter of fact, we here recall that quantum-kinematic ladder operators have already been used, rather successfully, in papers devoted to the quantum kinematic theory of the simple harmonic oscillator (Krause 1986) and to Galilean quantum kinematics (Krause 1988), where they played a prominent role. However, a general theory for these boson operators of non-Abelian non-compact Lie groups was still missing in the quantum kinematic formalism. This paper is devoted to filling this gap.

It is indeed well known that several physically relevant Lie algebras can arise very naturally as bilinear products of boson annihilation and creation operators (Lipkin 1966). Nonetheless, it is interesting to remark that in the present theory, once a non-compact (non-Albelian) Lie group has been quantized, the Lie algebras that may be generated by taking bilinear products of the kinematic boson operators appear as embedded in the quantized structure of the chosen group. They are intrinsic to this structure, and therefore no direct nor semidirect products are necessary to bring them into the fore. So they may play an important role in the classification of multiplets within the quantum kinematic models afforded by the quantization of the group (Krause 1986, 1988). This fact makes the present theory particularly interesting.

Although these ladder operators belong to the generalized enveloping quantum-kinematic algebra of the group (Krause 1993), this notion will not be used in this paper, because one can introduce the ladder operators of noncompact Lie groups in a direct fashion, quite independent of this new (i.e. unfamiliar) general notion.

The organization of this paper is as follows. Section 2 contains a rather sketchy review of the group quantization procedure and includes a discussion of some features of the basic quantum-kinematic invariant operators and of the (closed) quantum-kinematic algebra of non-compact Lie groups, as will be needed in the sequel. Next, in section 3, we apply these formalisms in order to build the complete set of r quantum-kinematic boson ladder operators, and we briefly discuss their main properties. Finally, section 4 includes a miscellaneous instance of a physical application, by calculating the three boson ladder operators of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$. Then, by considering only two of them, the Lie algebra of SU(1, 1) is obtained as an example of a Lorentz invariant internal structure, within the $\mathcal{P}_+^\uparrow(1,1)$ quantum kinematic theory. Although the aim of this paper is purely 'instrumental' for mathematical physics, concerning a new group-theoretic method of quantization, we end up in section 5 presenting a very general conjecture on some physical possibilities of the quantum-kinematic approach to the Poincaré group in the realm of elementary particle physics.

2. Non-Abelian quantum kinematics revisited

Here some of the main concepts leading to group quantization and non-Abelian quantum kinematics of non-compact Lie groups are repeated, because this new formalism is not known to most physicists. It is the intention to describe here (without proof) only those features which are relevant for the discussion of the quantum kinematic ladder operators.

The notation used throughout this paper is the same as paper I. Henceforth, G denotes a non-compact, connected and simply connected, r-dimensional non-Abelian Lie group (as, for instance, the universal covering group of a non-compact Lie group). Furthermore, we shall assume that there exists a coordinate patch $q=(q^1,\ldots,q^r)$ which covers the whole group manifold $M(G)$ and maintains everywhere a one-to-one correspondence with the elements of G; i.e. the coordinates q^a, $a=1,\ldots,r$ are real and provide a set of r essential parameters of G. This is a strong condition, to be sure. However, most Lie groups of physical interest are of a type known as 'linear Lie group', in the sense that they have at least one faithful finite-dimensional representation. It is well known that the whole of a connected linear Lie group of dimension r can be parametrized by r real numbers $q^1,\ldots,q^r$, which form a connected set in R^r. Of course, there is no requirement in general that this global parametrization of G be faithful. Nevertheless, there are many instances of non-compact, connected and simply connected linear Lie groups (of physical relevance) for which the global parametrization provides a one-to-one faithful mapping. For the sake of simplicity, this paper deals exclusively with Lie groups which satisfy this condition.

In the following $\bar{q}=\bar{q}(q)$ denotes that point in $M(G)$ which labels the inverse element corresponding to q, and $e=(e^1,\ldots,e^r)\in M(G)$ labels the identity element. Of course, $M(G)$ carries an analytic mapping, $g:M(G)\times M(G)\rightarrow M(G)$, that is endowed with the group property of G. Hence, in this parametrization one has a well defined set of r group-multiplication functions, $g^a(q';q)=q''^a\in M(G)$, which realize the group law in $M(G)$. (As a good general reference for these details, see Racah 1965.)

One defines Lie's (right and left) vector fields as follows:

$$X_a(q)\equiv R_a^b(q)\partial_b \qquad Y_a(q)\equiv L_a^b(q)\partial_b \tag{2.1}$$

where R_a^b and L_a^b are the (right and left) transport matrices for contravariant vectors in $M(G)$ which are obtained from $g^a(q';q)$ in the usual 'classical' fashion; i.e. $R_a^b(q)=\partial'_a g^b(q';q)|_{q'=e}$, and $L_a^b(q)=\partial'_a g^b(q;q')|_{q'=e}$. The Lie operators satisfy the Lie algebra

$$[X_a(q),X_b(q)]=f_{ab}^c X_c(q) \qquad [Y_a(q),Y_b(q)]=-f_{ab}^c Y_c(q)$$
$$[X_a(q),Y_b(q)]=0 \tag{2.2}$$

where the structure constants are given by $f_{ab}^c=R_{b,a}^c(e)-R_{a,b}^c(e)$.

In the forthcoming formalism we also need the inverse transport matrices in $M(G)$, which are defined by $\bar{R}_a^b(q)=\partial'_a(q';\bar{q})|_{q'=q}$, and $\bar{L}_a^b(q)=\partial'_a g^b(\bar{q};q')|_{q'=q}$. Clearly, one has: $R_a^b(e)=L_a^b(e)=\delta_a^b$, and $\bar{R}_a^c(q)R_c^b(q)=\bar{L}_a^c(q)L_c^b(q)=\delta_a^b$). As a matter of fact, the following 'mixed' transport matrices in $M(G)$ correspond to the adjoint representation G_A of G (cf paper I):

$$A_a^b(q)=R_a^c(q)\bar{L}_c^b(q) \qquad \bar{A}_a^b(q)=L_a^c(q)\bar{R}_c^b(q) \tag{2.3}$$

since one has

$$A^c_a(q')A^b_c(q) = A^b_a[g(q';q)] \tag{2.4}$$

and

$$A^b_a(e+\delta q) = \delta^b_a + \delta q^c f^b_{ca}. \tag{2.5}$$

Next, in order to *quantize the group* G let us associate the essential parameters q^a with a set of r commuting Hermitian operators Q^a, which act within the carrier space of the regular representation and may be interpreted as generalized 'position' operators of the group manifold (Krause 1985). Thus, within the common (rigged) Hilbert space $\mathscr{H}(G)$ that carries both (left and right) regular representations (paper I) one defines the following spectral integrals over the group manifold:

$$Q^a = \int \mathrm{d}\mu_L(q)|q\rangle_L q^a \langle q|_L = \int \mathrm{d}\mu_R(q)|q\rangle_R q^a \langle q|_R \tag{2.6}$$

i.e. one sets $Q^a = Q^a_L = Q^a_R$. The Qs are generalized position operators of $M(G)$, acting in $\mathscr{H}(G)$; in fact, one has $Q^{a\dagger} = Q^a$, $[Q^a, Q^b] = 0$, and

$$Q^a|q\rangle_L = q^a|q\rangle_L \qquad Q^a|q\rangle_R = q^a|q\rangle_R. \tag{2.7}$$

Hence, the Qs provide a complete set of commuting Hermitian operators in $\mathscr{H}(G)$. Here the Hurwitz invariant measures have been used on $M(G)$: $\mathrm{d}\mu_L(q) \equiv \mu_0 \bar{L}(q)\,\mathrm{d}^r q$, and $\mathrm{d}\mu_R(q) \equiv \mu_0 \bar{R}(q)\,\mathrm{d}^r q$, where $\bar{L}(q) = \det[\bar{L}^b_a(q)]$ and $\bar{R}(q) = \det[\bar{R}^b_a(q)]$. (In order to simplify the notation, it is assumed that $\mu_L = \mu_R = \mu_0$, but this choice is not strictly necessary.) (Cf the appendix in paper I for a unified formalism of the two regular representations which shall be used as theoretical frame in what follows.)

In paper I it was shown how the set of r basic quantum-kinematic invariant operators arise as a consequence of this group quantization approach to non-Abelian quantum kinematics. In fact, they correspond essentially to the generators of the right (left) regular representation acting as invariant operators within the left (right) regular representation of G. This feature is possible if one 'quantizes' the group (i.e. $q^a \to Q^a$), because in this fashion, and only in this fashion, the basic quantum-kinematic invariant operators appear as linear combinations of the generators, whose matrix coefficients are functions of the generalized position operators Q^a of G. Indeed, it was found that in the left regular representation (for instance) the invariant operators are given by

$$R_a(Q;L) = R^\dagger_a(Q;L) = \bar{A}^b_a(Q)L_b - \tfrac{1}{2}\mathrm{i}\hbar f^b_{ab} \tag{2.8}$$

where the Ls are the generators, $\bar{A}^b_a(q) = A^b_a(\bar{q})$ are the entries of the inverse matrix of the adjoint representation G_A, of G, and f^b_{ab} denotes a contraction of the structure constants. Indeed, from the (left) *generalized Heisenberg commutation relations* associated with G, namely (cf Krause 1985)

$$[Q^a, Q^b] = 0 \tag{2.9}$$

$$[Q^a, L_b] = i\hbar R^a_b(Q) \tag{2.10}$$

and from the (left) Lie algebra

$$[L_a, L_b] = -i\hbar f^c_{ab} L_c \tag{2.11}$$

it follows (paper I)

$$[R_a(Q;L), L_b] = 0 \qquad a, b = 1, \ldots, r. \tag{2.12}$$

In (2.10), we have defined

$$R_a^b(Q) = \int \mathrm{d}\mu_L(q)|q\rangle_L R_a^b(q)\langle q|_L \tag{2.13}$$

and (as was already said) the L_as are the generators of the left regular representation; i.e.

$$U_L(e+\delta q) = I - \left(\frac{i}{\hbar}\right)\delta q^a L_a \tag{2.14}$$

$$L_a|q\rangle_L = \mathrm{i}\hbar X_a(q)|q\rangle_L. \tag{2.15}$$

One next obtains the (closed) *quantum-kinematic algebras* of G, since from (2.10) one has

$$[F(Q), L_a] = \mathrm{i}\hbar X_a(Q)F(Q) \tag{2.16}$$

where $X_a(q)F(q) = R_a^b(q)F_{,b}(q)$, and therefore $X_a(q)A_b^c(q) = f_{ab}^d A_d^c(q)$ yields immediately

$$[A_b^c(Q), L_a] = \mathrm{i}\hbar f_{ab}^d A_d^c(Q). \tag{2.17}$$

In the same way, from $Y_a(q)A_b^c(q) = f_{ad}^c A_b^d(q)$ (cf also (3.6)), one gets

$$[A_b^c(Q), R_a] = \mathrm{i}\hbar f_{ad}^c A_b^d(Q) \tag{2.18}$$

where the Rs denote the generators of the right regular representation (that is, the kinematic invariants (2.8) of the left regular representation). Thus we see how non-Abelian quantum-kinematic commutation relations may be closed separately to form two finite-dimensional Lie algebras, which are committed in an interesting fashion with the quantized version of the matrices carrying the adjoint representation of the group. Equations (2.11) and (2.17) exhibit the left quantum kinematic algebra of G. (One obtains the right quantum kinematic algebra of G in a similar way.)

This finite closed structure embeds G into some larger Lie group G_{QK}, the *quantum kinematic group* of G which seems worthy of further investigation. One can identify, for instance, G_{QK} with the universal covering group associated to the quantum kinematic algebra of G. A few interesting comments concerning this structure are not out of place here. The (left) quantum-kinematic algebra of G yields the Lie algebra $\mathrm{g}_{QK}^{(L)}$ of the (left) quantum kinematic group $G_{QK}^{(L)}$, whose generators belong to the linear space defined by $L \cup A = \{L_a, A_b^c(Q);\ a, b, c = 1, \ldots, r\}$. Now, given that $[L, L] \subset L$, $[A, L] \subset A$, $[A, A] = \{0\}$, we note that the Lie algebra of G is a subalgebra of $\mathrm{g}_{QK}^{(L)}$, while $A = \{A_a^b(Q)a, b, = 1, \ldots, r\}$ corresponds to a commutative ideal of $\mathrm{g}_{QK}^{(L)}$. Hence, for the kind of Lie groups considered in this paper the Lie algebra $\mathrm{g}_{QK}^{(L)}$ is neither simple nor semisimple. In order to obtain the dimensions of $\mathrm{g}_{QK}^{(L)}$, one also has to recall that not all the adjoint matrix-operators $A_a^b(Q)$ are independent, because they have to satisfy the following r linear constraints:

$$f_{bc}^b A_a^c(Q) = f_{ba}^b \tag{2.19}$$

in general, and furthermore, in the applications, it may happen that some entries of the adjoint matrix are zeroth. So $\mathrm{g}_{QK}^{(L)}$ can be (at most) a r^2-dimensional Lie algebra. No stringent extra conditions, nor restrictions, on G appear to be necessary in this *mathematical construct*.

For proofs and more details concerning these results, the reader is referred to paper I. The main features concerning our interest here are shown in equations (2.8) and (2.12). Similar results hold for the right regular representation. Henceforth, for the sake of concreteness, let us work only within the left regular representation of G.

3. Canonical quantum-kinematic ladder operators

As an interesting application of the general formalism of quantum kinematics let us now consider the possibility of having a set of r first-order linear operators of the general form

$$\hat{a}_a(Q; L) = A_a(Q) + \mathrm{i}B_a^b(Q)L_b \tag{3.1a}$$

$$\hat{a}_a^\dagger(Q; L) = A_a(Q) - \mathrm{i}L_b B_a^b(Q) \tag{3.1b}$$

endowed with the following fundamental commutation properties:

$$[\hat{a}_a, \hat{a}_b] = 0 \qquad [\hat{a}_a, \hat{a}_b^\dagger] = \delta_{ab} \tag{3.2}$$

for $a, b, = 1, \ldots, r$. It will be proven here that such a set of boson ladder operators exists, notwithstanding the fact that G is a non-Abelian Lie group (of the special non-compact kind introduced in section 2). Furthermore, since $A_a(q)$ and $B_a^b(q)$ must be real regular functions of the qs everywhere on $M(G)$, it will also be proven that these operators are unique (within the addition of arbitrary constant multiples of the identity). Moreover, one can calculate them explicitly for any given non-compact Lie group of the assumed kind. In this way, one gets a complete set of non-Hermitian ladder operators acting in the Hilbert space that carries the regular representation of G, whose eigenvectors can be found, quite generally, as a system of special functions defined in $M(G)$.

First, we show the structure of the problem at hand. Taking into account the generalized Heisenberg commutation relations associated with G (cf (2.9)-(2.11)), as well as the definitions of Lie's vector fields acting on $M(G)$ (given in (2.1)), a straightforward calculation yields the following system of coupled nonlinear differential equations for the coefficients of the ladder operators:

$$B_a^c(q)X_c(q)A_b(q) - B_b^c(q)X_c(q)A_a(q) = 0 \tag{3.3}$$

$$B_a^d(q)X_d(q)B_b^c(q) - B_b^d(q)X_d(q)B_a^c(a) + f_{de}^c B_a^d(q)B_b^e(q) = 0 \tag{3.4}$$

$$B_a^c(q)X_c(q)A_b(q) + B_b^c(q)X_c(q)A_a(q) - \hbar B_a^c(q)X_c(q)X_d(q)B_b^d(q) = \hbar^{-1}\delta_{ab} \tag{3.5}$$

for all $q \in M(G)$. These are necessary and sufficient conditions for the operators defined in (3.1) to be endowed with the desired commutation relations (i.e. (3.2)). Of course, we are interested only in those solutions $A_a(q)$ and $B_a^b(q)$ that are regular everywhere on the group manifold, so that $|\langle\psi|\hat{a}_a|\psi\rangle| = |\langle\psi|\hat{a}_a^\dagger|\psi\rangle|$ remain finite for all $|\psi\rangle \in \mathscr{H}(G)$.

Now, in order to solve this rather formidable problem, one uses an indirect method, assuming any kind of admissible essential parameters q^a for G in $M(G)$. Furthermore, let us first present the set of generalized canonical ladder operators of G in terms of the invariant operators R_a (instead of using the generators L_a, as above); i.e. let us calculate the operators $\hat{a}_a(Q; R)$, say. Then we shall prove that $\hat{a}_a(Q, R) = \hat{a}_a(Q; L)$ holds, for all essential parametrizations of G.

To this end, we recall the generalized Heisenberg commutation relations for the generators of the right regular representation of G (paper I):

$$[Q^a, R_b] = \mathrm{i}\hbar L_b^a(Q). \tag{3.6}$$

As we know, within the 'left' working frame, the Rs are given in (2.8) and the entries $L_b^a(Q)$ have the spectral representations

$$L_b^a(Q) = \int \mathrm{d}\mu_L(q)|q\rangle_L L_b^a(q)\langle q|_L. \tag{3.7}$$

Hence, since the Qs commute, we have

$$[Q^a, \bar{L}_b^c(Q)R_c] = [Q^a, R_c\bar{L}_b^c(Q)] = \mathrm{i}\hbar\delta_b^a \tag{3.8}$$

quite generally. Thus, if we define the Hermitian operators

$$P_a(Q;R) = P_a^\dagger(Q;R) = \tfrac{1}{2}\{\bar{L}_a^b(Q)R_b + R_b\bar{L}_a^b(Q)\} = \tfrac{1}{2}[\bar{L}_a^b(Q), R_b]_+ \tag{3.9}$$

we can write down the standard Heisenberg commutation relations

$$[Q^a, P_b] = \mathrm{i}\hbar\delta_b^a \tag{3.10}$$

even if the parameters are not canonical and notwithstanding the fact that the non-compact G is a non-Abelian Lie group.

It is well known that the only way for (3.10) to be consistent with the Jacobi identity is that the Ps commute among themselves (since the Qs commute, cf (2.9)). In the present formalism, this requirement constitutes a challenge that must be proved. In order to tackle this problem let us first use

$$Y_b(q)\bar{L}_a^b(q) = [\ln \bar{R}(q)]_{,a} \tag{3.11}$$

(which can be proved in a direct manner) so that the Ps can be cast in the following form:

$$P_a = \bar{L}_a^b(Q)R_b - \tfrac{1}{2}\mathrm{i}\hbar[\ln \bar{R}(Q)]_{,a}. \tag{3.12}$$

(In order to help the reader at this point, note that (3.11), as well as $X_b(q)\bar{R}_a^b(q) = [\ln \bar{L}(q)]_{,a}$, hold quite generally, for they are immediate consequences of the formula $X_b(q)L_c^a(q) = Y_c(q)R_b^a(q)$ that follows from the group property of $g^a(q';q)$.) Then, since

$$[F(Q), R_a] = \mathrm{i}\hbar Y_a(Q)F(Q) \tag{3.13}$$

one gets

$$[\bar{L}_a^c(Q)R_c, \bar{L}_b^d(Q)R_d] = \mathrm{i}\hbar\{f_{cd}^e\bar{L}_a^c(Q)\bar{L}_b^d(Q) - \bar{L}_{b,a}^e(Q) + \bar{L}_{a,b}^e(Q)\}R_e = 0 \tag{3.14}$$

because, from the Lie algebra, one obtains

$$\partial_a\bar{L}_b^e(q) - \partial_b\bar{L}_a^e(q) = f_{cd}^e\bar{L}_a^c(q)\bar{L}_b^d(q). \tag{3.15}$$

Therefore, it follows

$$\begin{aligned}[P_a, P_b] &= \tfrac{1}{2}\mathrm{i}\hbar\bar{L}_b^c(Q)[R_c, \{\ln \bar{R}(Q)\}_{,a}] - \tfrac{1}{2}\mathrm{i}\hbar\bar{L}_a^c(Q)[R_c, \{\ln \bar{R}(Q)\}_{,b}] \\ &= \tfrac{1}{2}\hbar^2\bar{L}_b^c(Q)Y_c(Q)\{\ln \bar{R}(Q)\}_{,a} - \tfrac{1}{2}\hbar^2\bar{L}_a^c(Q)Y_c(Q)\{\ln \bar{R}(Q)\}_{,b}\end{aligned} \tag{3.16}$$

i.e. one has

$$[P_a, P_b]=0 \tag{3.17}$$

as required.

In this way, from equations (3.10) and (3.12), one defines the generalized ladder operators

$$\hat{a}_a(Q;R)=\frac{1}{\sqrt{2}}\left\{[Q^a+\tfrac{1}{2}Y_b(Q)\bar{L}_a^b(Q)]+\frac{\mathrm{i}}{\hbar}\bar{L}_a^b(Q)R_b\right\} \tag{3.18a}$$

$$\hat{a}_a^\dagger(Q;R)=\frac{1}{\sqrt{2}}\left\{\left[Q^a-\tfrac{1}{2}Y_b(Q)\bar{L}_a^b(Q)\right]-\frac{\mathrm{i}}{\hbar}\bar{L}_a^b(Q)R_b\right\} \tag{3.18b}$$

which certainly satisfy the commutation relations (3.2).

We now easily prove that if one defines the following operators

$$\hat{a}_a(Q;L)=\frac{1}{\sqrt{2}}\left\{[Q^a+\tfrac{1}{2}X_b(Q)\bar{R}_a^b(Q)]+\frac{\mathrm{i}}{\hbar}\bar{R}_a^b(Q)L_b\right\} \tag{3.19a}$$

$$\hat{a}_a^\dagger(Q;L)=\frac{1}{\sqrt{2}}\left\{[Q^a-\tfrac{1}{2}X_b(Q)\bar{R}_a^b(Q)]-\frac{\mathrm{i}}{\hbar}\bar{R}_a^b(Q)L_b\right\} \tag{3.19b}$$

maintaining the same q-parametrization of G, then one gets

$$\hat{a}_a(Q;L)\equiv\hat{a}_a(Q;R) \tag{3.20}$$

and furthermore $\hat{a}_a^\dagger(Q;L)$ is indeed the Hermitian adjoint of $\hat{a}_a(Q;L)$, for $a=1,\ldots,r$. To this end, all one needs to prove is that

$$\bar{L}_a^b(Q)R_b+R_b\bar{L}_a^b(Q)=\bar{R}_a^b(Q)L_b+L_b\bar{R}_a^b(Q). \tag{3.21}$$

This is the case in fact, because $f_{ab}^b=f_{bc}^c\bar{A}_a^b(q)$ holds for G_A (see paper I), and therefore (2.8) can be written in the forms

$$R_a=\bar{A}_a^b(Q)(L_b-\tfrac{1}{2}\mathrm{i}\hbar f_{bc}^c)=(L_b+\tfrac{1}{2}\mathrm{i}\hbar f_{bc}^c)\bar{A}_a^b(Q) \tag{3.22}$$

thus, using the definitions (2.3), equation (3.21) follows. This proves (3.20). Hence, it does not matter whether one uses the left or the right generators of the regular representation of G to define these canonical ladder operators of the group.

The task of obtaining the general form of the simultaneous eigenvectors $|\boldsymbol{n}\rangle$, as well as the generalized coherent states $|z\rangle$, associated with these canonical 'annihilation' and 'creation' boson operators of G (in particular, the discussion of their possible connections (if any) with Perelomov's generalized coherent states (Perelomov 1986)) will be considered elsewhere (Krause 1993).

The several (compact and non-compact) Lie algebras one may obtain by means of sets of bilinear combinations $\{\hat{a}_a\hat{a}_b, \hat{a}_a^\dagger\hat{a}_b, \hat{a}_a^\dagger\hat{a}_b^\dagger\}$ of r boson ladder operators are well known (Lipkin 1966, Barut and Raczka 1977). However, as was already mentioned in the Introduction, it is here underlined that these algebras are related with G in an intrinsic fashion, and therefore no direct nor semidirect product $\mathrm{SU}\times G$ (say) is required in order to obtain them, because they stem from the quantization of G. Thus, if G is a physically relevant Lie group, they may play an interesting role. (See, for instance, Krause 1986, 1988.)

In this fashion, we have found that

$$A_a(q)=\frac{1}{\sqrt{2}}\{q^a+\tfrac{1}{2}X_b(q)\bar{R}_a^b(q)\} \qquad B_a^b(q)=\frac{1}{\sqrt{2}}\hbar^{-1}\bar{R}_a^b(q) \tag{3.23}$$

yield the desired solution to equations (3.3)–(3.5). (One can prove this result quite generally, within any admissible parametrization of G.) Note that, for any given $|\psi\rangle \in \mathscr{H}(G)$, one has

$$\langle\psi|\hat{a}_a|\psi\rangle = \frac{1}{\sqrt{2}} \int \mathrm{d}\mu_L(q)\psi_L^*(q)\{q^a + \tfrac{1}{2i}[X_b(q)\bar{R}_a^b(q)] + \bar{R}_a^b(q)X_b(q)\}\psi_L(q) \tag{3.24a}$$

$$\langle\psi|\hat{a}_a^\dagger|\psi\rangle = \frac{1}{\sqrt{2}} \int \mathrm{d}\mu_L(q)\psi_L^*(q)\{q^a - \tfrac{1}{2i}[X_b(q)\bar{R}_a^b(q)] - \bar{R}_a^b(q)X_b(q)\}\psi_L(q) \tag{3.24b}$$

where one defines $\psi_L(q) = {}_L\langle q|\psi\rangle$. The fact that these ladder operators are the only admissible solution to equations (3.3)–(3.5), follows because the As and the Bs must be regular everywhere (in particular, at $q=0$), and because consistency demands $\langle\psi|\hat{a}_a|\psi\rangle^* = \langle\psi|\hat{a}_a^\dagger|\psi\rangle$. In this sense the boson ladder operators for G shown in (3.18) or (3.19) are essentially the unique solution of the problem.

Finally, an obvious (albeit important) fact is underlined. In order for the quantum kinematic boson operator to have the right dimensions, one should write

$$\hat{a}_a(Q;R) = \frac{1}{\sqrt{2}}\left(\frac{1}{\lambda_a}Q^a + \mathrm{i}\frac{\lambda_a}{\hbar}P_a\right) \qquad \hat{a}_a^\dagger(Q;R) = \frac{1}{\sqrt{2}}\left(\frac{1}{\lambda_a}Q^a - \mathrm{i}\frac{\lambda_a}{\hbar}P_a\right) \tag{3.25}$$

instead of (3.18), where the Ps are given in (3.12). The λs are real c-numbers, with chosen dimensions such that the $\hat{a}$s are dimensionless (i.e. purely numerical) operators, as they must be. This remark is important because in the applications, on physical grounds, one may interpret $\{\lambda_a, a=1,\ldots,r\}$ as a set of r phenomenological parameters which specify some properties of a physical system (like fixed Compton wavelengths, for instance, or otherwise, depending on the dimensions or units of the Qs). Nevertheless, since in this paper we shall not work out a concrete specific model of a system, here we set $\lambda_a = 1 (1 \leqslant a \leqslant r)$ for the sake of simplicity.

4. Ladder operators of the Poincaré group in two-dimensional spacetime

As an interesting application of the formalism of quantum-kinematic ladder operators, let us consider the group $\mathscr{P}_+^\uparrow(1,1)$ of Poincaré transformations in two-dimensional Minkowski spacetime:

$$\begin{aligned} x'^0 &= \gamma(q^2)(x^0 - q^2x^1) + q^0 \\ x'^1 &= \gamma(q^2)(x^1 - q^2x^0) + q^1 \end{aligned} \tag{4.1}$$

where $\gamma(q^2) = [1-(q^2)^2]^{-1/2}$. In the present parametrization, the group manifold is given by $M = \{-\infty < q^0 < +\infty, -\infty < q^1 < +\infty, -1 < q^2 < +1\}$, and $e = (0,0,0)$. The group law reads

$$\begin{aligned} q''^0 &= g^0(q';q) = q'^0 + \gamma(q'^2)(q^0 - q'^2q^1) \\ q''^1 &= g^1(q';q) = q'^1 + \gamma(q'^2)(q^1 - q'^2q^0) \\ q''^2 &= g^2(q';q) = (q'^2 + q^2)(1 + q'^2q^2)^{-1}. \end{aligned} \tag{4.2}$$

Hence, one has

$$R_a^b(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -q^1 & -q^0 & \gamma^{-2} \end{bmatrix} \qquad L_a^b(q) = \begin{bmatrix} \gamma & -\gamma q^2 & 0 \\ -\gamma q^2 & \gamma & 0 \\ 0 & 0 & \gamma^{-2} \end{bmatrix} \tag{4.3}$$

wherefrom one gets the familiar generators in $M(\mathscr{P}_+^\uparrow)$

$$X_0=\partial_0 \qquad X_1=\partial_1 \qquad X_2=-q^1\partial_0-q^0\partial_1+\gamma^{-2}\partial_2 \tag{4.4}$$

$$Y_0=\gamma(\partial_0-q^2\partial_1) \qquad Y_1=\gamma(\partial_1-q^2\partial_0) \qquad Y_2=\gamma^{-2}\partial_x \tag{4.5}$$

which satisfy the well known Lie algebra

$$[X_0,X_1]=0 \qquad [X_0,X_2]=-X_1 \qquad [X_1,X_2]=-X_0 \tag{4.6}$$

$$[Y_0,Y_1]=0 \qquad [Y_0,Y_2]=Y_1 \qquad [Y_1,Y_2]=Y_0 \tag{4.7}$$

and moreover

$$[X_a,Y_b]=0 \qquad a,b=0,1,2. \tag{4.8}$$

Let us then briefly review some features of the quantum kinematic theory of $\mathscr{P}_+^\uparrow(1,1)$, which will be needed for obtaining the desired ladder operators. The group $\mathscr{P}_+^\uparrow(1,1)$ is unimodular (i.e. $R(q)=L(q)=\gamma^{-2}$), so one defines the (left and right) Hurwitz measure:

$$\mathrm{d}\mu(q)=\gamma_0\gamma^2(q^2)\,\mathrm{d}q^0\,\mathrm{d}q^1\,\mathrm{d}q^2. \tag{4.9}$$

Hence, the position operators of $\mathscr{P}_+^\uparrow$ are given by

$$Q^a=\mu_0\iint \mathrm{d}q^0\,\mathrm{d}q^1\int_{-1}^{1}\mathrm{d}q^2\,\gamma^2(q^2)|q^0,q^1,q^2\rangle q^a\langle q^0,q^1,q^2| \tag{4.10}$$

for $a=0,1,2,$ where

$$\langle q'^0,q'^1,q'^2|q^0,q^1,q^2\rangle=\mu_0^{-1}\gamma^{-2}(q^2)\delta(q'^0-q^0)\delta(q'^1-q^1)\delta(q'^2-q^2) \tag{4.11}$$

and where (in this particular case) we have defined $|q^0,q^1,q^2\rangle_L=|q^0,q^1,q^2\rangle_R=|q^0,q^1,q^2\rangle$, once for all. In fact, the adjoint represention is defined by

$$\bar{A}_a^b(q)=\begin{bmatrix}\gamma & -\gamma q^2 & 0\\ -\gamma q^2 & \gamma & 0\\ q^1 & q^0 & 1\end{bmatrix}=A_a^b(\bar{q}) \tag{4.12}$$

so that $A(q)=R(q)\bar{L}(q)=1$ follows.

In this fashion, we obtain the Lie algebra of $\mathscr{P}_+^\uparrow$, which now reads

$$[L_0,L_1]=0 \qquad [L_0,L_2]=\mathrm{i}\hbar L_1 \qquad [L_1,L_2]=\mathrm{i}\hbar L_0 \tag{4.13}$$

as well as the generalized Heisenberg commutators of the left quantum-kinematic model of $\mathscr{P}_+^\uparrow$, given by

$$[Q^0,L_0]=\mathrm{i}\hbar \qquad [Q^1,L_0]=0 \qquad [Q^2,L_0]=0 \tag{4.14a}$$

$$[Q^0,L_1]=0 \qquad [Q^1,L_1]=\mathrm{i}\hbar \qquad [Q^2,L_1]=0 \tag{4.14b}$$

$$[Q^0,L_2]=-\mathrm{i}\hbar Q^1 \qquad [Q^1,L_2]=-\mathrm{i}\hbar Q^0 \qquad [Q^2,L_2]=\mathrm{i}\hbar\gamma^{-2}(Q^2). \tag{4.14c}$$

We note that the commutators obtained in (4.14*a*) and (4.14*b*) are canonical (as they must be indeed), while the commutation relations shown in (4.14*c*) are new.

The Lie algebra (4.13) has just one traditional invariant operator; namely, the 'mass square' Casimir operator

$$W=L_0^2-L_1^2. \tag{4.15}$$

In the physical interpretation of the formalism, this invariant operator yields the Klein–Gordon equation: $W|\psi_m\rangle = m^2c^2|\psi_m\rangle$ (*which corresponds to a super-selection* rule in $\mathscr{H}(\mathscr{P}_+^\uparrow)$). The important point to remark is that in the traditional approach to $\mathscr{P}_+^\uparrow(1,1)$ one obtains the theory of the Klein–Gordon equation, and nothing else.

However, the complete realm of the quantum-kinematic theory of $\mathscr{P}_+^\uparrow(1,1)$ is much broader than that of the traditional theory, because (once the group has been quantized) by means of the adjoint representation (4.12) one obtains three basic invariant operators, instead of only one. In effect, within the left regular representation of $\mathscr{P}_+^\uparrow$, these are given as follows:

$$R_0(Q;L) = \gamma(Q^2)(L_0 - Q^2L_1) \tag{4.16a}$$

$$R_1(Q;L) = \gamma(Q^2)(L_1 - Q^2L_0) \tag{4.16b}$$

and

$$R_2(Q;L) = Q^1L_0 + Q^0L_1 + L_2. \tag{4.16c}$$

Interestingly, the invariant operators R_μ, $\mu = 0, 1$, are obtained from the two-momentum operators L_μ by means of a quantized Lorentz transformation; i.e., $R_\mu = \Lambda_\mu^\nu(Q^2)L_\nu$. On the other hand, R_2 appears as a kind of total pseudo-Euclidean 'angular momentum' operator, related to the Lorentz hyperbolic 'rotation' in the two-dimensional Minkowski plane. Of course, these operators satisfy the right Lie algebra of $\mathscr{P}_+^\uparrow$. Moreover, one has

$$W = L_0^2 - L_1^2 \equiv R_0^2 - R_1^2. \tag{4.17}$$

Hence, one can 'diagonalize' this scheme in several ways. For instance, either (a) using the fact $[R_0, R_1] = 0$, or else (b) using $[W, R_2] = 0$. One thus reduces the left regular representation of $\mathscr{P}_+^\uparrow$(by means of the corresponding superselection rules) into physically meaningful Hilbert subspaces, which one hopes to interpret properly. The quantum kinematic invariants of $\mathscr{P}_+^\uparrow$ are first-order differential operators, and furthermore the formalism of $\mathscr{P}_+^\uparrow$ quantum-kinematics is automatically relativistic. (Work is in progress concerning this most interesting quantum-kinematic toy model: the super-selection rules $\{R_0, R_1\}$ yield the quantum-kinematic theory of the Dirac equation in $(1+1)$ dimensions; while, of course, $\{W, R_2\}$ yield the Klein–Gordon theory.)

After these prolegomena, we are ready to proceed with the ladder operator formalism of $\mathscr{P}_+^\uparrow(1,1)$. The inverse transport matrices of the group are given by

$$\bar{R}_a^b(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma^2q^1 & \gamma^2q^0 & \gamma^2 \end{bmatrix} \qquad \bar{L}_a^b(q) = \begin{bmatrix} \gamma & \gamma q^2 & 0 \\ \gamma q^2 & \gamma & 0 \\ 0 & 0 & \gamma^2 \end{bmatrix}. \tag{4.18}$$

Hence $\bar{L}(q) = \det[\bar{L}_a^b(q)] = \gamma^2(q^2)$ yields $[\ln \bar{L}(q)]_{,\mu} = 0$, for $\mu = 0, 1$, and $[\ln \bar{L}(q)]_{,2} = 2\gamma^2(q^2)q^2$; therefore from (3.19), after a few manipulations, we obtain

$$\hat{a}_0 = \frac{1}{\sqrt{2}}\left(Q^0 + \frac{\mathrm{i}}{\hbar}L_0\right) \qquad \hat{a}_1 = \frac{1}{\sqrt{2}}\left(Q^1 + \frac{\mathrm{i}}{\hbar}L_1\right) \tag{4.19}$$

which are quite familiar indeed (cf. (4.14)) and we also get the operators

$$\hat{a}_2 = \frac{1}{\sqrt{2}}Q^2 + \frac{1}{\sqrt{2}}\gamma^2(Q^2)\left(Q^2 + \frac{\mathrm{i}}{\hbar}R_2\right) \tag{4.20a}$$

$$\hat{a}_2^\dagger = \frac{1}{\sqrt{2}}Q^2 - \frac{1}{\sqrt{2}}\gamma^2(Q^2)\left(Q^2 + \frac{\mathrm{i}}{\hbar}R_2\right) \tag{4.20b}$$

which are non-trivial. In effect, these operators satisfy the following commutation relations:

$$[\hat{a}_0, \hat{a}_1]=0 \qquad [\hat{a}_0, \hat{a}_2]=0 \qquad [\hat{a}_1, \hat{a}_2]=0 \tag{4.21a}$$

$$[\hat{a}_0, \hat{a}_1^\dagger]=0 \qquad [\hat{a}_0, \hat{a}_2^\dagger]=0 \qquad [\hat{a}_1, \hat{a}_2^\dagger]=0 \tag{4.21b}$$

$$[\hat{a}_0, \hat{a}_0^\dagger]=I \qquad [\hat{a}_1, \hat{a}_1^\dagger]=I \qquad [\hat{a}_2, \hat{a}_2^\dagger]=I \tag{4.21c}$$

as the reader can check.

As we have already remarked, by taking bilinear products of these intrinsic ladder operators (in the well known manner) one can construct interesting operators that satisfy for instance the SU(2) and SU(3) algebras. The transformation law of the intrinsic generators of these SU(n) algebras ($n=2,3$) under (homogeneous and inhomogeneous) two-dimensional Lorentz transformations are quite involved, however, and they should be analysed carefully. Anyhow, they do not yield (in a direct fashion) Lorentz invariant intrinsic Lie algebras which could bear an immediate physical interpretation in the present model. Here we shall leave this subject as an open problem.

Nevertheless, there is an instance of an intrinsic Lorentz invariant algebra that arises rather naturally within the quantum-kinematic theory of $\mathscr{P}_+^\uparrow(1,1)$; this is the algebra of the group SU(1, 1). To see this feature, we have to consider the kinematics of $\hat{a}_0$ and $\hat{a}_1$. In order to obtain the kinematics of the canonical ladder operators, in general, one needs to use the following transformation laws:

$$U_L^\dagger(q)Q^a U_L(q)=g^a(q;Q) \tag{4.22}$$

$$U_L^\dagger(q)L_a U_L(q)=A_a^b(q)L_b \tag{4.23}$$

where the Us denote the unitary operators of the left regular representation of the group. In this fashion, for the operators $\hat{a}_0$ and $\hat{a}_1$ (we keep $\hat{a}_2$ out of the game) one obtains the following kinematic laws under the (left) action of $\mathscr{P}_+^\uparrow(1,1)$:

$$U_L^\dagger(q)\hat{a}_0 U_L(q)=\gamma(q^2)(\hat{a}_0-q^2\hat{a}_1^\dagger)+\frac{1}{\sqrt{2}}q^0 \tag{4.24a}$$

$$U_L^\dagger(q)\hat{a}_1 U_L(q)=\gamma(q^2)(\hat{a}_1-q^2\hat{a}_0^\dagger)+\frac{1}{\sqrt{2}}q^1. \tag{4.24b}$$

Thus, if one defines the operators

$$\hat{b}^\mu=(\hat{a}_0, \hat{a}_1^\dagger) \qquad \mu=0,1 \tag{4.25}$$

it follows

$$U_L^\dagger(q)\hat{b}^\mu U_L(q)=\Lambda_\nu^\mu(q^2)\hat{b}^\nu+\frac{1}{\sqrt{2}}q^\mu \tag{4.26}$$

where $\Lambda_\nu^\mu(q^2)$ is the 2×2 Lorentz matrix. This shows that $\hat{b}^\mu$ and $\hat{b}^{\mu\dagger}$ behave as contravariant vector-operators under homogeneous Lorentz transformations (i.e. $q^\mu=0$). This result makes the theory of the operators $\hat{b}_\mu$ and $\hat{b}_\mu^\dagger$ an interesting endeavour. For instance, we can define the following set of three Lorentz invariant operators within the $\mathscr{P}_+^\uparrow(1,1)$ quantum kinematic model:

$$K_0=\tfrac{1}{4}(\hat{b}^\mu\hat{b}_\mu^\dagger+\hat{b}_\mu^\dagger\hat{b}^\mu)=\tfrac{1}{2}(\hat{a}_0^\dagger\hat{a}_0-\hat{a}_1^\dagger\hat{a}_1) \tag{4.27a}$$

$$K_+=\tfrac{1}{2}\hat{b}^{\mu\dagger}\hat{b}_\mu^\dagger=\tfrac{1}{2}(\hat{a}_0^\dagger\hat{a}_0^\dagger-\hat{a}_1\hat{a}_1) \tag{4.27b}$$

$$K_-=\tfrac{1}{2}\hat{b}^\mu\hat{b}_\mu=\tfrac{1}{2}(\hat{a}_0\hat{a}_0-\hat{a}_1^\dagger\hat{a}_1^\dagger) \tag{4.27c}$$

and thus a few algebraic steps (cf (4.21)) yield

$$[K_0, K_+] = K_+ \qquad [K_0, K_-] = -K_- \qquad [K_+, K_-] = -2K_0 \tag{4.28}$$

which is precisely the well known Lie algebra SU(1, 1), often used in particle physics.

Now, without going into the possible physical interpretation of the multiplets belonging to this internal symmetry SU(1, 1) algebra, it seems interesting to discuss the structure of the SU(1, 1) multiplets pertaining to this model. We next briefly develop this issue.

It is well known that the quadratic operator (Perelomov 1986)

$$\boldsymbol{K}^2 = K_0^2 - \tfrac{1}{2}[K_+, K_-]_+ \tag{4.29}$$

is the Casimir operator of the Lie algebra (4.28). In the present case, this invariant operator may be cast in the following manifestly Lorentz scalar form

$$\boldsymbol{K}^2 = \tfrac{1}{4}(\varepsilon^{\mu\nu}\hat{b}^\dagger_\mu \hat{b}_\nu + I)(\varepsilon^{\lambda\rho}\hat{b}^\dagger_\lambda \hat{b}_\rho - I) \tag{4.30}$$

where $\varepsilon^{\mu\nu}$ is the two-dimensional Levy–Civitá symbol (i.e. $\varepsilon^{00} = e^{11} = 0$, $e^{01} = -\varepsilon^{10} = 1$). Since $\boldsymbol{K}^2$ is not a multiple of the identity, it follows that the representations we are going to build in this model are not irreducible, *sensu stricto.* However, as we shall see, they correspond in a simple manner to one of the two equivalent discrete series of representations of the SU(1, 1) algebra. Indeed, let us solve the eigenvalue problems

$$\boldsymbol{K}^2|k;\nu;\psi\rangle = \tfrac{1}{4}(k^2-1)|k,\nu,\psi\rangle \tag{4.31a}$$

$$K_0|k,\nu;\psi\rangle = \tfrac{1}{2}(k+2\nu+1)|k,\nu;\psi\rangle \tag{4.31b}$$

where the eigenvalue spectra will be given by: $k = 1, 2, 3 \ldots$ and $\nu = 0, 1, 2, \ldots$; and where ψ denotes a remaining degree of freedom in the definition of the eigenkets $|k,\nu;\psi\rangle \in \mathscr{H}(\mathscr{P}^\uparrow_+)$. (Our notation differs slightly from the usual one (cf Perelomov 1986) in a simple manner; i.e. here we set $|k,\nu\rangle \equiv |\kappa,\nu\rangle$ (of the standard notation), and we take $k = 2\kappa - 1$ (with $\kappa = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$), where μ is defined as $\mu = \kappa + \nu$ (with $\nu = 0, 1, 2 \ldots$). Of course, this change of notation is recommended by a mere inspection of the form of the Casimir operator in (4.30).)

Now, the explicit forms of $\boldsymbol{K}^2$ and K_0, as differential operators acting on (q^0, q^1) are obviously rather formidable, so we follow a different approach in order to solve (4.31). First, we observe that

$$|n_0, n_1;\psi\rangle = \int \mathrm{d}\mu(q)\phi_{n_0}(q^0)\phi_{n_1}(q^1)\psi(q^2)|q\rangle \tag{4.32}$$

are the most general solutions of

$$\hat{a}^\dagger_0 \hat{a}_0 |n_0, n_1;\psi\rangle = n_0|n_0, n_1;\psi\rangle \tag{4.33a}$$

$$\hat{a}^\dagger_1 \hat{a}_1 |n_0, n_1;\psi\rangle = n_1|n_0, n_1;\psi\rangle \tag{4.33b}$$

such that one has the orthogonality relations

$$\langle n_0', n_1';\psi|n_0, n_1;\psi\rangle = \delta_{n_0'n_0}\delta_{n_1'n_1}(\psi,\psi) \tag{4.34}$$

with

$$(\psi,\psi) = \mu_0 \int_{-1}^{1} \mathrm{d}q^2\, \gamma^2(q^2)|\psi(q^2)|^2 < \infty. \tag{4.35}$$

Furthermore, it is also easy to see that the set of eigenvectors $\{|n_0, n_1; \psi\rangle; (\psi, \psi) < \infty\}$ is a 'quasicomplete' basis, in the following sense: given any $|\psi\rangle \in \mathscr{H}(\mathscr{P}_+^\uparrow)$, one can always find a function $\hat{\psi}_{n_0 n_1}(q^2)$ such that

$$|\psi\rangle = \sum_{n_0} \sum_{n_1} |n_0, n_1; \hat{\psi}_{n_0 n_1}\rangle. \tag{4.36}$$

Indeed, $\hat{\psi}_{n_0 n_1}$ is unique, and it is given by

$$\hat{\psi}_{n_0 n_1}(q^2) = \int\int \mathrm{d}q^0 \, \mathrm{d}q^1 \, \phi_{n_0}(q^0) \phi_{n_1}(q^1) \langle q^0, q^1, q^2 | \psi\rangle \tag{4.37}$$

so that

$$\langle \psi | \psi \rangle = \sum_{n_0} \sum_{n_1} (\hat{\psi}, \hat{\psi})_{n_0 n_1} < \infty \tag{4.38}$$

necessarily holds.

Thus, let us write, without loss of generality

$$|k, \nu\rangle = \sum_{n_0} \sum_{n_1} D_{n_0 n_1 \cdot k\nu} |n_0, n_1\rangle. \tag{4.39}$$

(In the following manipulations we omit the 'ψ-paraphernalia'.) In this fashion, using the expression (4.27a) for K_0, one obtains almost immediately

$$|k, \nu\rangle = \sum_{n_1} E_{n_1 \cdot k\nu} |n_1 + k + 2\nu + 1, n_1\rangle \tag{4.40}$$

(since $n_0 \neq n_1 + k + 2\nu + 1 \Rightarrow D_{n_0 n_1 \cdot k\nu} = 0$). Next we use the expression

$$\varepsilon^{\mu\nu} \hat{b}_\mu^\dagger \hat{b}_\nu = a_0 a_1 - a_0^\dagger a_1^\dagger \tag{4.41}$$

to obtain

$$(\varepsilon^{\mu\nu} \hat{b}_\mu^\dagger \hat{b}_\nu \pm I)|n_0, n_1\rangle$$
$$= (n_0 n_1)^{1/2} |n_0 - 1, n_1 - 1\rangle - ((n_0+1)(n_1+1))^{1/2} |n_0 + 1, n_1 + 1\rangle \pm |n_0, n_1\rangle. \tag{4.42}$$

Then a rather lengthy (albeit standard) calculation yields the desired discrete series representations of the SU(1, 1) Lie algebra within $\mathscr{H}(\mathscr{P}_+^\uparrow)$; namely, one obtains

$$|k, \nu\rangle = N_{k\nu} \sum_{n_1=0}^{\infty} [n_1!(k + 2\nu + n_1 + 1)!]^{-1/2} d_{k\nu}^{n_1} |n_1 + k + 2\nu + 1, n_1\rangle \tag{4.43}$$

where the coefficients $d_{k\nu}^{n_1}$, have to satisfy the recursion relation

$$d_{k\nu}^{n+1} = n(n + k + 2\nu + 1) d_{k\nu}^{n-1} + k d_{k\nu}^n \tag{4.44}$$

with $d_{k\nu}^0 = 1$, and $N_{k\nu}$ is a normalization constant which we have defined as follows:

$$N_{k\nu} = (k + 2\nu + 1)!)^{1/2} E_{0.k\nu}. \tag{4.45}$$

Finally, we leave to the reader the task of introducing the 'ψ-dependence' of these eigenkets, in order to obtain the most general expression of the $|k, \nu; \psi\rangle$ eigenvectors.

This concludes the work on the canonical ladder operators and the SU(1, 1) intrinsic Lorentz invariant Lie algebra of $\mathscr{P}_+^\uparrow(1, 1)$ quantum kinematics in this paper.

5. Concluding remarks

In equations (4.27) and (4.28) we face a nice example of an intrinsic Lorentz invariant non-compact Lie algebra, arising within the $\mathscr{P}_+^\uparrow(1,1)$ quantized theory, which could be interpreted as an internal symmetry of the model, without recourse to any $SU(1,1)\times\mathscr{P}_+^\uparrow(1,1)$ scheme whatsoever (whether direct or semidirect). Of course, the physical interpretation of the multiplets belonging to this internal symmetry SU(1, 1) algebra (i.e. equations (4.43)–(4.45)), requires the previous knowledge of the quantum-kinematic theory of $\mathscr{P}_+^\uparrow(1,1)$ acting as the group of external symmetries of elementary systems in flat spacetime; that is, it requires the previous quantum-kinematic deductions of the Dirac equation and of the Klein–Gordon equation, as well as of their propagation kernels, in $(1+1)$-dimensions. This task will be tackled in forthcoming papers.

This intrinsic SU(1, 1) symmetry structure may play an interesting role in the 'elementary particle' toy models one expects to obtain from quantizing the Poincaré group in $(1+1)$-dimensions. For instance, as is indeed well known from the history of contemporary elementary particle physics (Dyson 1966), the year 1965 opened with various attempts at a theory incorporating a non-trivial (semi-direct) product of the group of internal symmetries and the Poincaré group of external symmetries for hadrons. (One thus expected to remove the mass degeneracy of the internal multiplets, as was the case of the Gell-Mann Okubo 'old-fashioned' SU(3) symmetry for strong interactions. The direct product is unable to perform this task.) SU(6), and even U(12), was favoured as the internal symmetry group of hadrons in most of these papers. These attempts, however, mixed internal SU(6) degrees of freedom and spin, in a similar manner as in Wigner's SU(4) theory of nuclear supermultiplets (Wigner 1937). Meanwhile, another series of papers appeared at the same time, in which severe mathematical inconsistencies were found in these theoretical attempts. Furthermore, rigorous theorems were proved to the effect that any such group combination must divide into a trivial direct product of the two (internal and external) symmetry structures. (See Dyson 1966, and papers reproduced therein.) Since then these ambitious working frames have been forgotten.

Of course, there have been several major developments in elementary particle physics since the 1960s; especially as consequences of the establishment of gauge field theories. (For more recent literature on the progress made in deriving or explaining the symmetries of the laws of nature in modern physics, see Froggatt and Nielsen 1991, and works reproduced therein.) We now have the standard model, and there seems to be general agreement about the origin of mass spectra in terms of spontaneous symmetry breakdown mechanism. Hence, the point to remark is that these developments put us in a rather tighter position today than was the case about 30 years ago. Moreover, many facets of these developments are well established experimentally, and therefore any new proposed theory must take them into account, and even attempt to go beyond them, rather than just replace them by another model. It is clear that these facts bring to the fore a lot of significant questions that would have to be answered before any new relativistic quantum formalism could arouse the interests of high-energy theoreticians today. This is plainly so.

Nevertheless, the importance must be stressed of the fact that *Lie groups can be quantized, in a mathematically consistent fashion and quite generally indeed*; i.e. whether they are Abelian or not, whether compact or non compact. This fact may become of the utmost importance for the future of quantum theory, because it throws new light upon that mysterious trick called 'quantization' (detaching it completely from many rather obscure philosophical questions, concerned mainly with the use and meaning of the 'correspondence principle') (Bacry 1988). Instead, this fundamental idea of quantum mechanics may be transformed into a well defined and general Lie group-theoretic notion. Thus, non-Abelian quantum kinematics (Krause 1985) offers a new perspective to look upon the notion of 'quantization', as well as to consider the very processes that have been used hitherto to build up reasonable quantum models of microphysical systems (Krause 1986, 1988). In this sense, it must be borne in mind that the usual 'canonical quantization' procedure is not a universal recipe (Komar 1971). As a matter of fact, under the present perspective, canonical quantization is nothing but a special case of Abelian quantum kinematics (as applied to the noncompact groups of rigid translations acting on Cartesian scaffoldings) (Weyl 1931).

In one way or another, these issues permeate all those rather successful models and/or mechanisms of today's high energy physics mentioned above. It is quite clear that one must not expect to answer all the pertinent questions (posed by these models) by just one stroke of luck. Rather, long and hard step-by-step work will be needed to this end. Nevertheless, there are several good reasons to expect that a complete and consistent non-Abelian quantum theory is possible to achieve, as a direct and simple group-theoretic generalization of the present quantum formalism. Moreover, it is perhaps urgently needed as a new theoretical tool in the real of elementary particle physics.

So as an instance of this possibility, and in the light of the results presented in this paper, it seems plausible to conjecture that the quantum-kinematic theory of the Poincaré group can bring under a completely new insight the old and most intriguing (albeit still unsolved) problem of the relations between the internal multiplets and the external relativistic symmetries of elementary particles and of the origin of their mass spectra. At the risk of being perhaps overly optimistic, the author deems this conjecture worthy of much further research.

Finally, it should be made clear that the interest of the generalized boson annihilation and creation operators, introduced in this paper, for Lie group theory and mathematical physics in general, is quite independent of this particular conjecture.

Acknowledgments

The author wishes to express his gratitude to Florin Constantinescu for his friendly hospitality and for many enlightening discussions during the period when much of this work was done. The hospitality received from J W Goethe Universität, Frankfurt am Main, Germany, is also acknowledged hereby. He also wishes to acknowledge Professor L O O'Raifeartaigh (Dublin Institute for Advances Studies) for a critical, and very helpful, reading of the manuscript.

This work was supported in part by DIUC and by FONDECYT, through Grants nos 0269/92 and 1930562/93.

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